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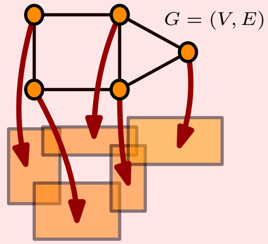
Seminar Discrete Structures
SS 2017

THE BOXICITY OF GRAPHS

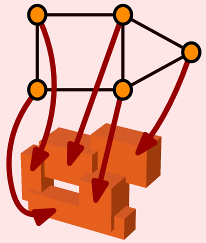
The concept of the Boxicity

$G = (V, E)$ a simple, undirected graph.

Box-representation of a graph: An assignment of each vertex $v \in V$ to an axis parallel box $\chi(v) \subset \mathbb{R}^d$, so that $\chi(v_1) \cap \chi(v_2) \neq \emptyset \Leftrightarrow \{v_1, v_2\} \in E$ for all $v_1, v_2 \in V$.



2-dimensional box-representation



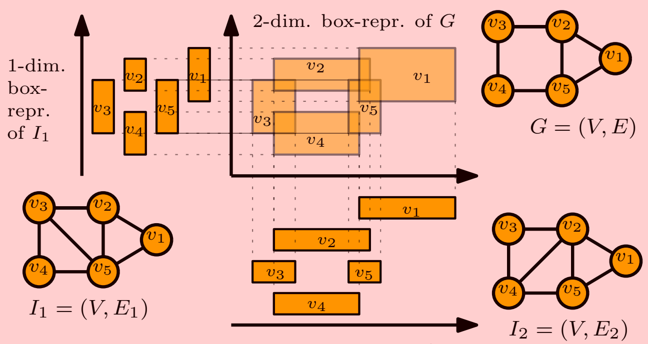
3-dimensional box-representation

Boxicity: Minimal dimension needed for a box representation. (Notation: box(G))

Equivalent: Smallest number n so that there are interval graphs $I_1, ..., I_n, I_i = (V, E_i)$ with $E = E_1 \cap ... \cap E_n$.

(Where an interval graph is a graph that has a 1-dimensional box-representation)

To see the equivalence:
By projection onto the axis, a n-dimensional box representation yields the n interval graphs $I_1, ..., I_n$ as needed.



$E = E_1 \cap E_2$

The concept of the Cubicity

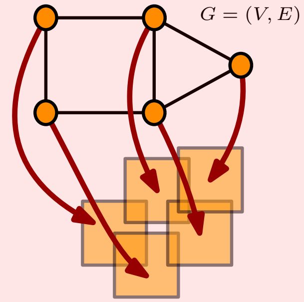
Cube-representation: A box-representation, where the boxes are cubes (equal lengths) and all have the same size.

Cubicity: Minimal dimension needed for a cube-representation. (Notation: cub(G))

Equivalent: Smallest number n so that there are unit interval graphs $I_1, ..., I_n, I_i = (V, E_i)$ with $E = E_1 \cap ... \cap E_n$.

(Where a unit interval graph is a graph that has a 1-dimensional cube-representation)

Clearly: $\text{box}(G) \leq \text{cub}(G)$



2-dimensional cube-representation

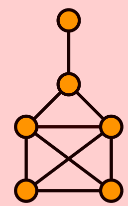
An upper bound on the boxicity

- Calculation of $\text{box}(G), \text{cub}(G)$ is NP-complete (*Cozzens*, 1981).
- *Roberts*, 1969: $|V| = n$, then $\text{cub}(G) \leq \lfloor \frac{2n}{3} \rfloor$, $\text{box}(G) \leq \lfloor \frac{n}{2} \rfloor$
- This implies: $\text{cub}(G), \text{box}(G) \in \mathcal{O}(n)$
- Simple examples of equality exist (*Roberts*, 1969).
- *Adiga, Chandran, Mathew* 2013: G k-degenerate, then
  $\text{cub}(G) \leq (k+2)\lceil 2e \cdot \log n \rceil$ (e eulerian number)
- They also provide this useful tool for general graphs $G = (V, E)$:
  $S \subset V$, $G[S]$ the subgraph induced by S, H the graph obtained from G by removing all edges in $G[S]$. Then:
  $\text{box}(G) \leq 2 \cdot \text{box}(H) + \text{box}(G[S])$
- By choosing S in a specific way so that H becomes k-degenerate *Esperet* (2015) deduces from the previous theorems:
  $\text{box}(G) \leq (15e + 1)\sqrt{m \cdot \log n}$ $(m = |E|, n = |V|)$
- This implies $\text{box}(G) \in \mathcal{O}(\sqrt{m \cdot \log m})$

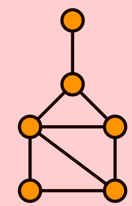
k-degenerate graphs

G **k-degenerate**: Every subgraph S contains a vertex v with degree $\deg(v) \leq k$ in S.

Examples:



3-degenerate



2-degenerate